Defects with Character:

Zero-Energy Majorana Modes in Condensed-Matter Systems

Tutorial talk by Bertrand I. Halperin,

Harvard University

Nano-QT 2016

Kyiv

October 9, 2016
Localized zero-energy Majorana modes

• Extra “zero energy” degrees of freedom that have been hypothesized to occur at isolated point defects in some special correlated-electron systems. (1D and 2D)

• Majorana modes should have some very peculiar properties, have generated much interest.

• So far, only modest evidence of their realization in experiments on actual physical systems.

• I will try to explain the concepts and what people hope to observe, and will comment on the experimental situation at the end.
Outline

• What are the peculiar properties of localized Majorana modes?
• Where might one find such modes?
• One proposal: Hybrid of a semiconductor nanowire with strong spin-orbit interactions, coupled to a superconductor.
• Network of “topological superconductor” wires connected by T-junctions. Manipulation of Majorana sites on such a network.
• Mathematical description of localized Majorana modes, and (non)-relation to Majorana fermions in particle physics.
• Condition for Majorana modes in a hybrid nanowire.
• Comments on the experimental situation.
Localized Majorana Modes

Defining Properties

The parent system, a system with no defects, no boundary, should have a finite energy gap for electronic excitations.

A system with $N$ well-separated point defects (Majorana sites) should have a “degenerate” ground state, such that the set of ground states forms a Hilbert space with dimension $2^{N/2}$.

Ground-state degeneracy is “robust” to local perturbations.
Magnetic impurities in an insulator have localized low-energy degrees of freedom. Simplest case has $S=1/2$; an isolated impurity has two degenerate states, “spin-up” and “spin-down.” System with $N$ impurities has low energy Hilbert space with dimension $2^N$. Basis states by specifying spin-up or spin-down at each site.

If no applied magnetic field, and no interactions between spins, states are degenerate in energy. Take into account exchange interactions, degeneracy is split. Exchange interactions fall off exponentially with distance for spins in an insulator, so splitting can be negligible if spins are far apart.
Comparison: Localized spins in an insulator

But degeneracy is not robust, even if spins are far apart. Spins have magnetic moment, observable quantity, couples to any local magnetic field. Magnetic moments give rise to dipole interaction between spins, falls off as $1/r^3$, not exponentially.
Majorana Modes

System with $N$ point defects (Majorana sites): Low energy Hilbert space has dimension $2^{N/2}$.

Effectively: one $S=1/2$ degree of freedom for every two Majorana sites.

If sites are far apart, no local observable can distinguish between states in the Hilbert space; energies are precisely degenerate. For finite separation $r$, energy splittings fall off exponentially:

$$E_{\text{split}} \propto e^{-r/\xi}.$$  ( $\xi \approx$ microscopic coherence length)

Splitting can be negligible if $r$ is sufficiently large, and no local perturbation can split this degeneracy.

In theoretical discussions, often assume that separations are sufficiently large that $E_{\text{split}}$ can be set equal to zero.
Manipulation of Majorana states

Suppose there is a way to physically move defects around “adiabatically” as a function of time. (Adiabatically means that motion is slow on the frequency scale defined by the lowest finite-energy excitations in the system, but fast on the scale of the exponentially small energy splittings of states in the “zero-energy” Hilbert space. Electron system will stay in low-energy Hilbert space.

If the final site positions are the same as the initial positions, except for possible interchanges of site positions, the final Hilbert space is identical to the original, and the result of the manipulation is described by a unitary transformation in the Hilbert space.

Unitary transformations do not commute. Final result depends on the order in which interchanges are performed and the way in which site-positions move around each other. But results are insensitive to any continuous deformations of the paths of motion, as long as sites remain far apart at all times.

In 2D systems, result depends only on the topology of the braiding operation. Adiabatic manipulations are said to be “Topologically Protected”
Topologically protected quantum computing?

It has been proposed that localized Majorana states could be useful for quantum computing (Kitaev 1990s)

Form qubits out of pairs of defects, perform quantum manipulations by moving them around each other. Result depends only on topology, so is “topologically” protected, provided that defects are far enough apart that exponential splittings are truly negligible. Protected manipulations do not access complete set of unitary transformations, must be supplemented with non-protected processes to build a universal quantum computer. Could still be very useful.

Huge obstacles in practice to using Majorana states for quantum computing.

But constructing and observing Majorana states would be very interesting in itself.
Where can we find systems with point defects that support localized Majorana states?

Fractional Quantized Hall State at filling factor 5/2. (Highly correlated 2D electron system in a strong magnetic field.)

Plateau in Hall conductance first observed by Willett et al in 1987. (In a very-high-quality GaAs heterostructure)

Wave function proposed by Moore and Read in 1991, implies that localized Majorana states are associated with the elementary charged quasiparticles, which have charge e/4.

Various numerical calculations suggest that the 5/2 state should be of the Moore-Read type, but experimental evidence is far from conclusive.
Majorana modes should occur at defects in some other 2D systems

Point vortices in 2D “Topological Superconductors”.

Various possible realizations have been proposed over the years: (Moore & Read 1991, Greiter et al. 1992, Volovik 1999, Read & Green 2000, Ivanov 2001.)
(Fu and Kane, PRL 2008)
(Fujimoto, 2010).
(Sau et al., PRL 2010; Alicea PRB 2010)

Experimental realizations are lacking.
Majorana modes can also occur at point defects in 1D Topological Superconductors

Defects with Majorana modes occur at the end of a TS, at a boundary between a TS and an ordinary superconductor or at a T-junction between two TS wires.

Proposed Realization:

Hybrid structure: Semiconductor nanowire with strong spin-orbit coupling, in applied magnetic field, in tunneling contact with a bulk s-wave superconductor. (Oreg et al.; Sau et al.; Potter and Lee, 2010)

Related ideas: Zero-energy state at a domain wall for one-dimensional Dirac fermions, by Jackiw and Rebbi in 1976.

Spin-polarized 1D wire coupled to a bulk p-wave superconductor. (Kitaev)
Superconductor should be narrower than the magnetic penetration depth (no Meissner effect) but wide enough so there are no random phase slips.
Cooper pairs tunnel into InAs, induce superconductivity by proximity effect. Gates control electron density in wire.
Networks of Hybrid Wires

Majorana sites on different wires can be interchanged or moved around each other by connecting wires together through T-junctions (or Y-junctions), and selectively depleting different segments of the wires, using gates.

“Topologically protected” manipulations can be done similar to vortices in a 2D topological superconductor.

[Alicea et al (Nat Phys 2011); Clarke, Sau, Tewari (PRB 2011); Halperin, Oreg, Stern, Refael, Alicea & von Oppen (PRB 2012)]
Architecture of a Simple Network

Three wire segments joined at a T- or Y- junction.

Top view.
Gates not shown.
Filled and empty segments

Wire segments can be filled or emptied by activation of selected gates.

Red segments are filled, in a topological superconductor regime.

Green segments are depleted of electrons.
Straight or bent wire segments have Majorana modes at ends
Segment with a T-junction has four Majorana modes
Elementary Manipulation A
Clockwise interchange of two Majoranas on the same wire segment
Elementary Manipulation A

Clockwise interchange of two Majoranas on the same wire segment
Elementary Manipulation A
Clockwise interchange of two Majoranas on the same wire segment
Elementary Manipulation A
Clockwise interchange of two Majoranas on the same wire segment
Elementary Manipulation A

Clockwise interchange of two Majoranas on the same wire segment
Elementary Manipulation A
Clockwise interchange of two Majoranas on the same wire segment
Elementary Manipulation B

Clockwise interchange of two Majoranas on adjacent wire segments
Elementary Manipulation B
Clockwise interchange of two Majoranas on adjacent wire segments
Elementary Manipulation B

Clockwise interchange of two Majoranas on adjacent wire segments
Elementary Manipulation B
Clockwise interchange of two Majoranas on adjacent wire segments
Elementary Manipulation B

Clockwise interchange of two Majoranas on adjacent wire segments

A  C
|   |
B  D
Canonical experiments require initialization, manipulation, measurement

**Initialization**: Change gate potentials to create superconducting wire segment out of vacuum.
Canonical experiments require **initialization, manipulation, measurement**

**Initialization**: Change gate potentials to create superconducting wire segment out of vacuum.
Canonical experiments require **initialization**, **manipulation**, **measurement**

**Initialization**: Change gate potentials to create superconducting wire segment out of vacuum.
Canonical experiments require initialization, manipulation, measurement

**Initialization**: Change gate potentials to create superconducting wire segment out of vacuum.

At low energies, only Cooper pairs can tunnel in or out of the bulk superconducting reservoir, so electron number parity of created segment must be even.
Canonical experiments require initialization, manipulation, measurement

**Measurement**: Change gate potentials to completely deplete superconducting wire segment.
Canonical experiments require **initialization, manipulation, measurement**

**Measurement**: Change gate potentials to completely deplete superconducting wire segment.
Canonical experiments require **initialization, manipulation, measurement**

**Measurement**: Change gate potentials to completely deplete superconducting wire segment.
Canonical experiments require **initialization, manipulation, measurement**

**Measurement**: Change gate potentials to completely deplete superconducting wire segment.

If wire segment has **odd electron parity**, then one electron will be left over when wire is depleted. This must be ejected into the bulk superconductor above its energy gap. This **requires extra energy** to be supplied by the gate voltage, which maybe measured in principle.

**Measurement** determines the **electron parity** of a wire segment.
Experiment 1

Two clockwise interchanges of Majoranas on adjacent wire segments ( ~ one clockwise $2\pi$ rotation )

Prepare

Interchange B&C

Interchange B&C

Measure
Experiment 2

(One clockwise interchange + one counterclockwise interchange)
Experiment 3

(Single interchange)

Prepare

Interchange B&C

Measure

Result: 50% + +

50% - -

Wires in coherent superposition of different parity states.
Meaning of coherent superposition

Phase difference should be independent of time, not sensitive to environment.

No difference in energy between states of even and odd number parity.

States cannot be distinguished by any local measurement

No difference in charge density
Isolated filled segments have a fixed electron parity (independent of time). To modify the parity of a wire segment, it is necessary to connect and disconnect several wire segments. Total electron parity is conserved during allowed manipulations.
General Result: 2

If wire segment has a definite electron parity, rotation or interchange of ends has no physical effect.

If wire is in a coherent superposition of two parity states, interchange or $2\pi$ rotation does have an effect. Opposite parity states acquire different phase factors.
Similar effects occur if phase of superconducting order parameter $\Delta$ in a wire segment is changed by $2\pi$. (Parity is preserved, but opposite parity states acquire different phase factors.)

This can happen if one moves a vortex in the bulk superconductor around the ends of the wire segment.
3D Networks

Can also generalize to 3-dimensional network of wires and T-junctions. Constraints: Rashba $E$-field is perpendicular to wire, and projection of $B$ on plane of wire and $E$ is not zero at any point.

Rules are slightly more complicated, but still topologically protected.
Mathematical description and relation to Majorana concept in elementary particle physics.
Mathematical Description of a Localized Majorana Mode

Peculiar “zero-energy” degree of freedom, bound to a defect, at a specified position $\mathbf{R}_i$.

Associated quantum operator $\gamma_i$ is “half of a normal fermion operator”.

For different sites, $\gamma_i \gamma_j = -\gamma_j \gamma_i$.

But $\gamma_i^2 = 1$, $\gamma_i^+ = \gamma_i$. Annihilation operator is the same as the creation operator. (“Particle is its own antiparticle.”)

Two Majorana operators can be combined to form a normal fermion operator: Define $c = (\gamma_1 + i \gamma_2) / 2$, $c^+ = (\gamma_1 - i \gamma_2) / 2$, then $c^2 = 0$, $\{c,c^+\} = 1$, $c^+c$ has eigenvalues 0, 1

$i \gamma_1 \gamma_2 = 2c^+c-1$ has eigenvalues 1, -1. (“parity of the pair”).

Energy does not depend on occupation $c^+c$ if defects are very far apart.
Majorana Concept in elementary particle physics

In 1937, Ettore Majorana realized that a neutral fermion can be its own antiparticle.

Hypothesized that it may apply to the neutrino.

Experimental test: Look for neutrino-less double beta decay: nucleus emits two electrons and no neutrinos. Possible only if neutrino is a Majorana fermion. (Two neutrinos are emitted virtually but annihilate each other before the electrons escape). Not yet seen, but observation could be very difficult. Event rates are expected to be very small, proportional to neutrino mass.
Big differences between a Majorana neutrino and a localized zero-energy mode in condensed matter

**Neutrino** is a *propagating* particle; very weakly interacting; no known way to physically produce a localized neutrino state.

Even if it were possible to produce a localized neutrino with zero energy, it would not have the special characteristics of an isolated Majorana mode in condensed matter. Because 3D Majorana particle has spin, would find an even number of localized Majorana modes at the same position in space. No non-abelian statistics.
How to get topological superconductivity in a hybrid nanowire
Single-mode InSb nanowire with strong Zeeman field

\[ H_1 = p^2 + V(x) + \alpha E \cdot p \sigma_y - B \cdot \sigma \]

Orientation of wire (w) chosen in x-direction. Rashba E-field, perpendicular w, chosen in z-direction. Magnetic field B should not be parallel to the y axis. \( \sigma \) is the electron spin.

Spectrum for \( B = 0 \)

Spectrum for \( B \neq 0 \)
Nanowire tunnel-coupled to superconductor

Bogoliubov-de Gennes \( \hat{H}_{\text{eff}} = \hat{H}_1 + \hat{H}_2 \)

\[
\hat{H}_1 = p^2 + V(x) + \alpha E_\text{F} p \sigma_y - \mathbf{B} \cdot \sigma
\]

\[
\hat{H}_2 = \int d\mathbf{r} \Delta_0^* \psi^\uparrow(\mathbf{r}) \psi^\downarrow(\mathbf{r}) + \text{h.c.}
\]

Adjust Fermi level to fall inside Zeeman gap at \( p = 0 \). Two Fermi-points remain at, \( p = \pm k_F \), with single spin state at each. \( \hat{H}_2 \) pairs electrons with opposite spin. \( \sigma_y \propto p/B \), has opposite sign at \( \pm k_F \); but \( \sigma_z > 0 \).

\( \hat{H}_2 \) induces a superconducting gap at Fermi points of magnitude

\[
|\Delta| \sim |\Delta_0| \frac{p}{B}.
\]

Gap will vanish if \( E_\text{F} \) is moved to a point where \( p=0 \) is at Fermi energy.
Diagonalizing the BdG Hamiltonian.

Find operators \( \gamma_j = \sum_\sigma \int d\mathbf{r} \left[ u_{j\sigma}(\mathbf{r}) \psi_\sigma(\mathbf{r}) + v_{j\sigma}(\mathbf{r}) \psi_\sigma^+(\mathbf{r}) \right] \),

Such that \([\gamma_j, H_{\text{eff}}] = E_j \gamma_j\).

Normalize such that \(H_{\text{eff}} = (1/2) \sum_j E_j \gamma_j^+ \gamma_j + \text{constant}\)

In general, energies occur in pairs:
For every \(E_j > 0\), exists \(E_m = -E_j\), with \(\gamma_m = \gamma_j^+\). So can restrict sum in \(H_{\text{eff}}\) to positive energy states and remove factor \((1/2)\)

Special case: Can have solutions with zero energy, with \(\gamma_j^+ = \gamma_j\), \(v_{j\sigma} = u_{j\sigma}^*\). (Majorana States)

Claim: For a long hybrid wire in the topological state, localized zero energy Majorana states occur at the two ends of the wire.
Robustness

An isolated zero-energy Majorana state is robust.

Since non-zero energy states must exist in pairs, no local perturbation can convert an isolated zero-energy state to a non-zero energy state.

A pair of Majorana modes can combine to form a pair of states with non-zero energy, but only if their wave functions overlap. Wavefunctions decay exponentially in space with decay length equal to the superconducting correlation length in the bulk of the wire. To lose the zero energy states one must either shorten the wire so the wavefunction overlap or one must change the bulk Hamiltonian such that the correlation length diverges. This requires a phase transition where the bulk energy gap goes to zero.

Existence of zero energy Majorana states at a wire end is a determined by Hamiltonian in the bulk, not by details of the wire end.
Experimental Evidence

Evidence for a Majorana state at the end of a hybrid superconductor-InSb nanowire in an applied magnetic field. [Mourik et al (Kouwenhoven group)]
Results

For $B=0$, low-bias tunneling conductance is $\sim 0$ below $V \sim 200 \mu eV$, interpreted as superconducting energy gap. For $B > 150$ mT, conductance peak at low voltage is interpreted as tunneling into a Majorana state at wire end.
Have localized Majorana modes been seen in experiments?

Electron states in a superconductor can always be written as a linear combination of two Majorana operators. Majorana modes always occur in pairs, and they can never be truly isolated in a finite wire.

The interesting question is: if one is tunneling into a low energy state arising from a Majorana mode at one of the wire, where is the partner Majorana located? Is it located at the opposite end of the superconducting segment, or is it perhaps associated with a defect or impurity inside the segment. Low energy does not guarantee large spatial separation.
Energy of a Majorana pair

Energy splitting of a Majorana pair with separation $x$ should go roughly as

$$E = E_0 \ e^{-x/\xi} \ \cos k_F x$$

where $\xi$ is the induced superconducting coherence length, order 200 nm, large compared to the Fermi wavelength. $E_0$ depends on details.

$E < 0 \ \iff \ \text{ground state has odd number of electrons.}$
Exponential Splitting of Majorana End Modes

From Albrecht et al, Nature 531, 206 (2016)  [Marcus Lab]
Recent Development

Experimental evidence for Topological Superconductivity and localized Majorana modes in a ferromagnetic chain of Fe atoms on the surface of superconducting Pb.

S. Nadj-Perge et al, arXiv:1410.0682 (Yazdani lab)
The existence of localized Majorana modes is clearly established in theoretical model systems. But they have not yet been clearly observed in experiments.

Hybrid superconductor-semiconductor wires are currently the most promising systems.

Majorana modes probably exist in the 5/2 Fractional Quantized Hall state, but experimental studies are still ambiguous.

If well-isolated Majoranas can be created and manipulated, they should have remarkable properties.